

# Direct and inverse scattering problem by an unbounded rough interface with buried obstacles

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## Abstract

In this paper, we consider the direct and inverse problem of scattering of time-harmonic waves by an unbounded rough interface with a buried impenetrable obstacle. We first study the well-posedness of the direct problem with a local source by the variational method; the well-posedness result is then extended to scattering problems induced by point source waves (PSWs) and hyper-singular point source waves (HSPSWs). For PSW or HSPSW incident waves, the induced total field admits a uniformly bounded estimate in any compact subset far from the source position. Moreover, we show that the scattered field due to HSPSWs can be approximated by the scattered fields due to PSWs. With these properties and a novel reciprocity relation of the total field, we prove that the rough surface and the buried obstacle can be uniquely determined by the scattered near-field data measured only on a line segment above the rough surface. The proof substantially relies upon constructing a well-posed interior transmission problem for the Helmholtz equation.

**Keywords:** Inverse scattering, unbounded rough interface, buried obstacle, variational method, reciprocity relation, interior transmission problem.

**MSC 2010:** 35R30, 78A46, 35Q60.

## 1 Introduction

This paper is concerned with the problem of scattering of time-harmonic waves from an unbounded rough interface with a buried impenetrable obstacle in two dimensions. This model problem has extensive applications in physics and engineering, such as ocean exploration by sonar and remote sensing by synthetic aperture radar (SAR). The *unbounded rough interface* is assumed to be a nonlocal perturbation of an infinite plane such that the interface lies within a finite distance of the original plane. We assume further that the whole space is separated by the unbounded rough interface with the medium above and below the rough interface being both homogeneous and isotropic. Many work has been done on the numerical approximation and computation for rough surface scattering problems (see, e.g. [30, 31, 27, 5, 13] and the references quoted therein). The mathematical theory of rough surface scattering problems has also been studied by many authors. For example, Chandler-Wilde and his co-workers studied the well-posedness of the direct rough surface scattering problems by integral equation methods (see, e.g.

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[34, 35, 7]). Recently, Chandler-Wilde and Monk [9] proved the well-posedness by the variational approach which can be applied to more general non-smooth surfaces and also gave the a priori estimate of the solution in terms of data with an explicit dependence on wave number.

We consider the direct scattering problem modeled by the Helmholtz equation  $\Delta u + k^2 u = g$  in  $\mathbb{R}^2$  with wave number  $k^2 = k_1^2$  above the rough surface and  $k^2 = k_2^2$  below it. And the total field  $u$  satisfies transmission conditions on rough surface and boundary conditions on the buried impenetrable obstacle  $D$ . The model includes the scattering caused by a local source when  $g \in L^2(\mathbb{R}^2)$  with compact support and point source wave or singular point source when  $g$  denotes a general distribution. Figure 1 demonstrates the geometrical set-up. To accomplish the scattering problem, a radiation condition at infinity is required. Due to the unbounded rough surface, the Sommerfeld radiation condition is no longer valid. We require that the solution above the rough surface and below the buried obstacle can be represented in an integral form as a superposition of upward (downward) propagating and evanescent plane waves. This radiation condition is equivalent to the upward propagating radiation condition firstly proposed by Chandler-Wilde and Zhang in [33].

Related work on the direct scattering problem can be found in [9, 22, 23]. They considered the acoustic scattering from sound-soft rough surface or penetrable rough layers and the electromagnetic scattering from rough layers with a absorbing medium by the variational method. Different from their work, this paper focuses on the study of wave scattering from an unbounded rough interface with a buried impenetrable obstacle. The existence of an obstacle in the model will make the analysis much more complicated. In particular, we can not obtain a priori estimate in terms of data in case of non-absorbing medium because of sign-changing terms on the boundary of the obstacle. Whereas, the a priori estimate can be established under the condition that the medium below the rough surface is absorbing. This condition fits well with some engineering applications, such as underground remote sensing since the soil is in fact energy-absorbing. In the non-absorbing case, the variational formulation is reduced into an operator equation with the operator being Fredholm with index zero. Thus, the existence of solutions will follow from the uniqueness of solutions. In particular, our scattering problem is well-posed in the case that the obstacle is partially coated in a non-absorbing medium. The existence of solutions to the scattering problem due to PSWs and HSPSWs was studied already in a different setting in [8]. However, we have the following key observations. First, we show that the total field is uniformly bounded with respect to the source positions in any compact set far from the source position (see Theorem 4.1). This uniform bound is useful for constructing a well-posed interior transmission problem that will be used to prove uniqueness of the inverse problem. Moreover, we show that the scattered field due to HSPSWs can be approximated by the scattered field due to PSWs (see Theorem 4.3). Since we will mainly employ the singularity of HSPSWs in the proof of the inverse scattering problem, this approximation result makes it possible to use the scattered field induced by PSWs instead.

As for the uniqueness of inverse scattering problems, there exist a vast literature on the bounded obstacle scattering problems (see, e.g. [21, 19, 24]). Moreover, inverse scattering from multilayered background medium is also studied, see [14, 25]. However, the method in [14] only works in the case that the transmission constant  $\lambda \neq 1$ , and the method in [25] relies heavily on a priori estimates of the scattering solution on the interface of the layered medium which is hard to be established in rough surface scattering problems. There are also numerous uniqueness results on inverse scattering problems on periodic structures, which can be viewed as a special case of rough surfaces (see, e.g. [2, 20, 16, 29]). Recently, scattering problems from a two-layered

background medium with planar interface are also considered in [26, 12, 28].

To the authors' knowledge, there are only few uniqueness results on inverse rough surface scattering problems. In [6], Chandler-Wilde and Ross proved that a sound-soft rough surface in a lossy medium can be determined uniquely by the scattered field data due to only one incident plane wave. And Hu [17] proved that sound-soft rough surfaces and rough layers (transmission constant  $\lambda \neq 1$ ) can be determined uniquely by the scattered field data due to PSWs.

Recently, Yang, Zhang and Zhang [32] proposed a new method to prove uniqueness of inverse scattering from penetrable obstacles which includes the case that transmission constant  $\lambda = 1$ . The main idea is based on proof by contradiction and is summarized as follows. First suppose that there are two obstacles which produces the same scattered data. One constructs a local well-posed interior transmission problem with boundary data given by the scattered field corresponding to point sources and different obstacles. The scattered field corresponding to one obstacle can be shown uniformly bounded as the points sources approaches the boundary of the other obstacle. Then one uses this fact and the well-posedness of the interior transmission problem to get the contradiction that the  $H^1$ -norm (or  $L^2$ -norm) of point sources (or hyper singular point sources) are uniformly bounded. An important feature of this idea is that the interior transmission problem is constructed locally. This motivates us to adapt similar idea to prove uniqueness results in rough surface scattering problem. We note that in the proof of uniqueness result in [32], a denseness result (Theorem 5.5 in [11]) of plane incident waves is used, which can not be generalized to point source incident waves in rough surface scattering problem. In our proof, the denseness is replaced by the approximation property of the scattered field due to PSWs and HPSWs.

This paper is organized as follows. In Sections 2 and 3, we formulate the boundary value problem modeling the direct scattering problem with a local source  $g$  and give its equivalent variational formulation. Then we study the solvability of the variational formation in two different cases according to whether the medium below the rough interface is lossy. In Section 4, we show that similar results also hold for PSWs and HSPSWs incident waves. Moreover, we prove two important results about the scattered fields, namely the uniform bound of the total fields with respect to the source positions and the approximation property about the scattered fields. In section 5, we first state some results on interior transmission problem and then we prove the uniqueness of the inverse scattering problem based on the interior transmission problem and a novel reciprocity relation.

## 2 The direct problem and its variational formulation

In this section, we present the direct problem and its equivalent variational formulation. To this end, we need some notations. For  $h \in \mathbb{R}$ , let  $\Gamma_h = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = h\}$  and denote  $U_h^\pm = \{x \in \mathbb{R}^2 \mid x_2 \gtrless h\}$ . For a given bounded function  $f \in C^2(\mathbb{R})$ , we define  $f_- := \inf_{x \in \mathbb{R}} f(x) > 0$ ,  $f_+ := \sup_{x \in \mathbb{R}} f(x) < +\infty$ . Then the rough interface is defined by  $\Gamma := \{(x_1, f(x_1)) \mid x_1 \in \mathbb{R}\}$ . Denote by  $D$  the buried impenetrable obstacle with boundary  $\partial D \in C^2$ , and assume that  $D$  is below the rough interface, this is,  $\text{dist}(\overline{D}, \overline{U_{f_-}^+}) > 0$ . For simplicity, we assume that  $D \subset U_0^-$ . Assume further that the buried obstacle is partially coated by a thin dielectric layer so that  $\partial D = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ , where  $\Gamma_1$  and  $\Gamma_2$  are two disjoint open subsets of  $\partial D$ . Denote by  $\Gamma_1$  the coated part with an impedance function  $\beta(x)$  and by  $\Gamma_2$  the uncoated part. In particular, the obstacle is sound-soft if  $\Gamma_1 = \emptyset$ , and fully coated obstacle (impedanced obstacle) corresponds to the case

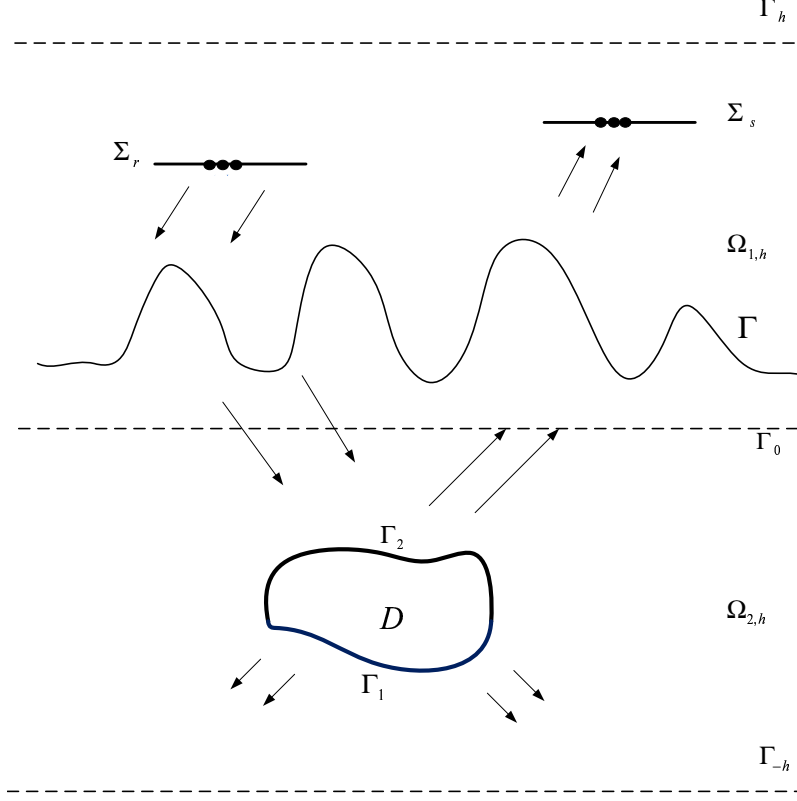


Figure 1: Scattering from an unbounded rough interface with impenetrable obstacles.

when  $\Gamma_2 = \emptyset$ . Note also that the obstacle becomes sound-hard when the impedance function  $\beta(x)$  vanishes on  $\partial D$ . Denote by  $\Omega_1$  the region above  $\Gamma$ , and by  $\Omega_2$  the region below  $\Gamma$  and outside  $D$ . We also define  $\Omega_{1,h} := \{x = (x_1, x_2) \in \Omega_1 | x_2 < h\}$ ,  $\Omega_{2,h} := \{x = (x_1, x_2) \in \Omega_2 | x_2 > -h\}$ . For simplicity, let  $h > f_+$  satisfy that  $D \subset U_{-h}^+$ , and define  $\Omega_h := \{x \in \mathbb{R}^2 | -h < x_2 < h\} \setminus \overline{D}$ . For  $M > 0$ , denote  $\Omega_{i,h}(M) := \{x \in \Omega_{i,h} | -M \leq x_1 \leq M\}$  and  $\gamma_i(\pm M) := \{x \in \Omega_{i,h} | |x_1| = \pm M\}$ ,  $i = 1, 2$ . Let  $\nu(x)$  be the unit normal vector at  $x \in \Gamma$  pointing into  $\Omega_1$  or at  $x \in \partial D$  pointing out of  $D$ . For  $\varepsilon > 0$ , and  $y \in \mathbb{R}^2$ , denote by  $B_\varepsilon(y)$  the ball centered at  $y$  with radius  $\varepsilon$ . We first consider the scattering problem with a local source  $g \in L^2(\mathbb{R}^2)$  compactly supported in  $\Omega_h$ . The cases with incident waves PSWs and HSPSWs will be considered in Section 4.

We are now ready to formulate the scattering problem. Assume that  $\Omega_1$  and  $\Omega_2$  are filled with two isotropic homogenous materials denoted by the wave numbers  $k_1$  and  $k_2$  respectively satisfying

$$\begin{aligned} k_1^2 > \operatorname{Re}(k_2^2) > 0 \text{ or } 0 < k_1^2 < \operatorname{Re}(k_2^2) \\ \operatorname{Im}(k_2^2) \geq 0 \end{aligned} \quad (2.1)$$

This means that the medium above the rough interface is non-absorbing and that below the interface it may be absorbing. The condition (2.1) is usually termed as *non-trap* condition because it ensures the uniqueness of the scattering problem. We only consider the case  $k_1^2 > \operatorname{Re}(k_2^2)$ ; the other one case can be dealt with similarly (see Remark 3.10). The total field  $u$  due to the source

$g$  satisfies the Hemholtz equations

$$\Delta u + k^2 u = g \quad \text{in } \mathbb{R}^2 \setminus D \quad (2.2)$$

where  $k^2(x) := k_1^2$  for  $x \in \Omega_1$  and  $k^2(x) := k_2^2$  for  $x \in \Omega_2$ . On the rough interface, the total field  $u$  satisfies the transmission condition

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu} \quad \text{on } \Gamma, \quad (2.3)$$

where  $u^+, \partial u^+ / \partial \nu$  (resp.  $u^-, \partial u^- / \partial \nu$ ) denote the limits on  $\Gamma$  from the above (resp. below). This implies that the field and its normal derivatives are continuous across the interface. On the boundary of the buried obstacle  $\partial D$ , the field  $u$  satisfies a mixed boundary condition

$$\frac{\partial u}{\partial \nu} u + i\beta u = 0 \quad \text{on } \Gamma_1, \quad u = 0 \quad \text{on } \Gamma_2 \quad (2.4)$$

with  $\beta \geq 0$ ,  $\beta \in C(\Gamma_1)$  representing the physical property of the obstacle. We use the condition  $\mathcal{B}(u) = 0$  to denote the boundary condition (2.4).

Since  $\Omega_1$  and  $\Omega_2$  are unbounded, radiation conditions at infinity must be imposed on the scattered and transmitted field. It is worth to note that the standard Sommerfeld radiation condition is not appropriate for rough surface scattering problems. Similar to [9], the scattered field is required to be represented in an integral form as a superposition of upward (resp. downward) propagating and evanescent plane waves in  $U_h^+$  (resp.  $U_{-h}^-$ ).

For  $\phi \in L^2(\mathbb{R})$ , define its Fourier transform by

$$\hat{\phi}(\xi) := \mathcal{F}\phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-ix_1 \cdot \xi) \phi(x_1) dx_1, \quad \xi \in \mathbb{R}.$$

We require  $u$  to satisfy the angular spectrum representation:

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(i[(x_2 - h)\sqrt{k_1^2 - \xi^2} + x_1 \cdot \xi]) \mathcal{F}(u|_{\Gamma_h})(\xi) d\xi, \quad x \in U_h^+, \quad (2.5)$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(i[-(x_2 + h)\sqrt{k_2^2 - \xi^2} + x_1 \cdot \xi]) \mathcal{F}(u|_{\Gamma_{-h}})(\xi) d\xi, \quad x \in U_{-h}^-, \quad (2.6)$$

where  $h > f_+$ ,  $u|_{\Gamma_h} \in L^2(\Gamma_h)$ , and the square root in the expression takes the negative imaginary axis as the branch cut in the complex plane, that is for  $z \in \mathbb{C}$ ,  $z = z_1 + iz_2$ ,  $z_1, z_2 \in \mathbb{R}$ , we have

$$\sqrt{z} = \text{sgn}(z_2) \sqrt{\frac{|z| + z_1}{2}} + i \sqrt{\frac{|z| - z_1}{2}} \quad (2.7)$$

Define  $V_h := \{u|u \in H^1(\Omega_h), u = 0 \text{ on } \Gamma_2\}$ . The inner product and norm in  $V_h$  are the same as the function space  $H^1(\Omega_h)$ . Then the direct scattering problem can be stated as the following boundary value problem.

**Boundary Value Problem (BVP):** Given a source  $g \in L^2(\mathbb{R}^2)$  compactly supported in  $\Omega_h$ , find  $u$  such that  $u \in V_h$  satisfying (2.2)-(2.4) and the radiation conditions (2.5) and (2.6).

For  $s \in \mathbb{R}$ , define  $H^s(\Gamma_h)$  as the completion of  $C_0^\infty(\Gamma_h)$  in the following norm

$$\|\phi\|_{H^s(\Gamma_h)}^2 := \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{\phi}(\xi)|^2 d\xi \quad (2.8)$$

Introduce Dirichlet-to-Neumann (DtN) operators  $T_1$  on  $\Gamma_h$  and  $T_2$  on  $\Gamma_{-h}$

$$(T_1\phi)(x_1) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k_1^2 - \xi^2} \exp(ix_1 \cdot \xi) \hat{\phi}(\xi) d\xi, \quad x \in \Gamma_h$$

$$(T_2\phi)(x_1) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{k_2^2 - \xi^2} \exp(ix_1 \cdot \xi) \hat{\phi}(\xi) d\xi, \quad x \in \Gamma_{-h}$$

The next lemma collects some properties of the DtN operators.

**Lemma 2.1.** (i)  $T_1 : H^{1/2}(\Gamma_h) \rightarrow H^{-1/2}(\Gamma_h)$  and  $T_2 : H^{1/2}(\Gamma_{-h}) \rightarrow H^{-1/2}(\Gamma_{-h})$  are bounded linear operators.

(ii) For  $\phi \in H^{1/2}(\Gamma_h)$  and  $\psi \in H^{1/2}(\Gamma_{-h})$ , we have

$$\operatorname{Re} \left( \int_{\Gamma_h} \bar{\phi} T_1 \phi ds \right) \leq 0, \quad \operatorname{Im} \left( \int_{\Gamma_h} \bar{\phi} T_1 \phi ds \right) \geq 0, \quad (2.9)$$

$$\operatorname{Re} \left( \int_{\Gamma_{-h}} \bar{\psi} T_2 \psi ds \right) \leq 0, \quad \operatorname{Im} \left( \int_{\Gamma_{-h}} \bar{\psi} T_2 \psi ds \right) \geq 0 \quad (2.10)$$

(iii) For  $\phi_j \in H^{1/2}(\Gamma_h)$  and  $\psi_j \in H^{1/2}(\Gamma_{-h})$ ,  $j = 1, 2$ , we have

$$\int_{\Gamma_h} \phi_1 T_1 \phi_2 ds = \int_{\Gamma_h} \phi_2 T_1 \phi_1 ds, \quad \int_{\Gamma_{-h}} \psi_1 T_2 \psi_2 ds = \int_{\Gamma_{-h}} \psi_2 T_2 \psi_1 ds. \quad (2.11)$$

*Proof.* (i) From the definition of  $T_1$  and  $T_2$  and by (2.8), we have

$$\|T_1 \phi\|_{H^{-1/2}(\Gamma_h)}^2 = \int_{\mathbb{R}} (1 + \xi^2)^{-1/2} |\widehat{T_1 \phi}(\xi)|^2 d\xi = \int_{\mathbb{R}} (1 + \xi^2)^{-1/2} |\sqrt{k_1^2 - \xi^2}| \hat{\phi}(\xi)|^2 d\xi$$

Noting that  $|\sqrt{k_1^2 - \xi^2}|^2 \leq C(k_1)(1 + \xi^2)^{1/2}$ , then

$$\|T_1 \phi\|_{H^{-1/2}(\Gamma_h)}^2 \leq C(k_1) \|\phi\|_{H^{1/2}(\Gamma_h)}^2$$

Similarly, we have

$$\|T_1 \psi\|_{H^{-1/2}(\Gamma_{-h})}^2 \leq C(k_2) \|\psi\|_{H^{1/2}(\Gamma_{-h})}^2$$

(ii) (2.9) is proved in [9]. For  $T_2$ , we have

$$\int_{\Gamma_{-h}} \bar{\psi} T_2 \psi ds = \int_{\mathbb{R}} i \sqrt{k_2^2 - \xi^2} |\mathcal{F}\psi(\xi)|^2 d\xi.$$

By (2.7) and note that  $\operatorname{Im} k_2 \geq 0$ ,

$$\operatorname{Re}(\sqrt{k_2^2 - \xi^2}) \geq 0, \quad \operatorname{Im}(\sqrt{k_2^2 - \xi^2}) \geq 0$$

which implies (2.10).

(iii) It is proved in [9] (see Lemma 3.2 therein). □

**Lemma 2.2.** (i) If  $u$  satisfies (2.5) with  $u|_{\Gamma_h} \in H^{1/2}(\Gamma_h)$ , then  $u \in H^1(U_h^+ \setminus U_a^+) \cap C^2(U_h^+)$ , for every  $a > h$ ,

$$\Delta u + k_1^2 u = 0 \quad \text{in } U_h^+,$$

$\gamma_+ u = u|_{\Gamma_h}$ , and

$$\int_{\Gamma_h} \bar{v} T_1 \gamma^+ u \, ds + k_1^2 \int_{U_h^+} u \bar{v} \, dx - \int_{U_h^+} \nabla u \cdot \nabla \bar{v} \, dx = 0, \quad v \in C_0^\infty(\mathbb{R}^2)$$

where  $\gamma^+$  is the trace operator from  $V_h$  to  $H^{1/2}(\Gamma_h)$ . Further, for all  $a > h$ , the restriction of  $u$  and  $\nabla u$  to  $\Gamma_a$  lies in  $L^2(\Gamma_a)$  and

$$\int_{\Gamma_a} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 + k_1^2 |u|^2 \right) ds \leq 2k_1 \operatorname{Im} \left( \int_{\Gamma_a} \bar{u} \frac{\partial u}{\partial x_2} ds \right). \quad (2.12)$$

Moreover, for all  $a > h$  (2.5) holds with  $h$  replaced by  $a$ .

(ii) If  $u$  satisfies (2.6) with  $u|_{\Gamma_{-h}} \in H^{1/2}(\Gamma_h)$ , then  $u \in H^1(U_{-h}^- \setminus U_a^-) \cap C^2(U_{-h}^-)$ , for every  $a < -h$ ,

$$\Delta u + k_2^2 u = 0 \quad \text{in } U_{-h}^-,$$

$\gamma_- u = u|_{\Gamma_{-h}}$ , and

$$\int_{\Gamma_{-h}} \bar{v} T_2 \gamma^- u \, ds + k_2^2 \int_{U_{-h}^-} u \bar{v} \, dx - \int_{U_{-h}^-} \nabla u \cdot \nabla \bar{v} \, dx = 0, \quad v \in C_0^\infty(\mathbb{R}^2)$$

where  $\gamma^-$  is the trace operator from  $V_h$  to  $H^{1/2}(\Gamma_{-h})$ . Further, for all  $a < -h$ , the restriction of  $u$  and  $\nabla u$  to  $\Gamma_a$  lies in  $L^2(\Gamma_a)$  and

$$\int_{\Gamma_a} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 + \operatorname{Re}(k_2^2) |u|^2 \right) ds \leq \left( 2(\operatorname{Re}(k_2^2))^{1/2} + \sqrt{2}(\operatorname{Im}(k_2^2))^{1/2} \right) \operatorname{Im} \left( \int_{\Gamma_a} \bar{u} \frac{\partial u}{\partial x_2} ds \right) \quad (2.13)$$

Moreover, for all  $a < -h$ , (2.5) holds with  $-h$  replaced by  $a$ .

*Proof.* The lemma can be proved similarly as in [9] and [22].  $\square$

Multiplying (2.2) by  $v \in V_h$  and integrating by parts, we can easily get the following equivalent variational formulation of the problem **(BVP)**.

Find  $u \in V_h$  such that

$$a(u, v) = - \int_{\Omega_h} g \bar{v} \, dx \quad \text{for all } v \in V_h. \quad (2.14)$$

where the sesquilinear form  $a(u, v)$  is defined as

$$a(u, v) := \int_{\Omega_h} (\nabla u \nabla \bar{v} - k^2 u \bar{v}) \, dx - \int_{\Gamma_h} \bar{v} T_1 u \, ds - \int_{\Gamma_{-h}} \bar{v} T_2 u \, ds - i \int_{\Gamma_1} \beta u \bar{v} \, ds. \quad (2.15)$$

The problem **(BVP)** and variational formulation (2.14) are equivalent in the following sense: Given  $u$  satisfying **(BVP)**,  $u|_{\Omega_h}$  is a solution of (2.14). Conversely, if  $u$  is a solution of (2.14), it is easy to see that  $u$  satisfies the transmission condition (2.3) on  $\Gamma$  and the boundary condition (2.4) on  $\partial D$ . From Lemma 2.2, we can expand  $u$  to  $\mathbb{R}^2$  by (2.5) and (2.6) with continuous traces on  $\Gamma_h$  and  $\Gamma_{-h}$ . Moreover,  $u$  satisfies  $\Delta u + k^2 u = g$  in the distribution sense, with  $g$  extended to be zero outside  $\Omega_h$ . Consequently,  $u$  also satisfies the problem **(BVP)**.

From the definition of  $a(\cdot, \cdot)$ , the boundedness of  $T_1$  and  $T_2$  and the fact that  $\beta \in C(\Gamma_1)$ , the sesquilinear form  $a(\cdot, \cdot)$  defined by (2.15) is bounded, that is,

$$|a(u, v)| \leq C \|u\|_{V_h} \|v\|_{V_h}, \quad u, v \in V_h$$

By the Riesz representation theorem there exists a bounded linear operator  $\mathcal{A}_k : V_h \rightarrow V_h^*$  such that

$$\langle \mathcal{A}_k u, v \rangle_{V_h} = a(u, v), \quad u, v \in V_h$$

where  $V_h^*$  denotes the dual space of  $V_h$  and  $\langle \cdot, \cdot \rangle_{V_h}$  is the dual pair between  $V_h^*$  and  $V_h$ . Note that  $\mathcal{A}_k$  depends on  $k$  since  $a(\cdot, \cdot)$  does. Therefore, the variational formulation (2.14) can also be simplified as the following operator equation

$$\mathcal{A}_k u = \mathcal{G} \tag{2.16}$$

where  $\mathcal{G} \in V_h^*$  is defined by  $\mathcal{G}(v) := - \int_{\Omega_h} g \bar{v} dx$ ,  $v \in V_h$ , with the local source  $g \in L^2(\mathbb{R}^2)$  and  $\|\mathcal{G}\|_{V_h^*} \leq \|g\|_{L^2(\mathbb{R}^2)}$ .

### 3 Well-posedness of the variational formulation

In this section, we prove the well-posedness of the variational problem (2.14) and the well-posedness of the problem **(BVP)** follows subsequently. The former mainly depends on the generalized Lax-Milgram theory of Babuška (Theorem 2.15 in [18]). Thus we reformulate the variational problem into a more general problem in the framework of functional analysis: given  $\mathcal{G} \in V_h^*$  find  $u \in V_h$  such that  $\mathcal{A}_k u = \mathcal{G}$  or

$$a(u, v) = \mathcal{G}(v), \quad \forall v \in V_h \tag{3.1}$$

**Theorem 3.1.** (*Generalized Lax-Milgram Theorem*) *Let  $H$  be a Hilbert space with norm and inner product given by  $\|\cdot\|$  and  $(\cdot, \cdot)$  respectively. Suppose that  $a : H \times H \rightarrow \mathbb{C}$  is a bounded sesquilinear form such that there holds the inf-sup condition*

$$\gamma := \inf_{0 \neq u \in H} \sup_{0 \neq v \in H} \frac{|a(u, v)|}{\|u\| \|v\|} > 0 \tag{3.2}$$

*and the transposed inf-sup condition*

$$\sup_{0 \neq u \in H} \frac{|a(u, v)|}{\|u\|} > 0.$$

*Then for each  $\mathcal{G} \in H^*$  there exists a unique solution  $u \in H$  such that*

$$a(u, v) = \mathcal{G}(v) \quad \forall v \in H, \quad \text{with } \|u\| \leq \gamma^{-1} \|\mathcal{G}\|_{H^*}.$$



The inf-sup condition, which is the key requirement in the Generalized Lax-Milgram Theorem, can be verified by the following lemma [18].

**Lemma 3.2.** *Suppose there exists  $C > 0$  such that for all  $u \in V_h$  and  $\mathcal{G} \in V_h^*$  satisfying (3.1) it holds that*

$$\|u\|_{V_h} \leq C \|\mathcal{G}\|_{V_h^*}. \quad (3.3)$$

*Then the inf-sup condition (3.2) holds with  $\gamma \geq C^{-1}$  and  $H = V_h$ .*

In order to obtain the a priori estimate (3.3), we consider two cases depending on whether or not the medium below the rough interface is absorbing.

### 3.1 Case 1: $0 < \operatorname{Re}(k_2^2) < k_1^2, \operatorname{Im}(k_2^2) > 0$

It is shown in Lemma 4.5 of [9] that the a priori estimate (3.3) for the solution of (3.1) can be obtained by the a priori estimate for the solution of (2.14) with  $g \in L^2(\Omega_h)$ . We now prove the later estimate by the Rellich identity technique, which was used in [9, 22].

**Lemma 3.3.** *For the given  $g \in L^2(\Omega_h)$ , let  $u \in V_h$  satisfy the problem*

$$a(u, v) = -(g, v) \text{ for all } v \in V_h. \quad (3.4)$$

*Then*

$$\|u\|_{V_h} \leq C \|g\|_{L^2(\Omega_h)} \quad (3.5)$$

*Proof.* Taking the real and imaginary part of (3.3) with  $v = u$  leads to the equation:

$$\int_{\Omega_h} (|\nabla u|^2 - k^2 |u|^2) dx - \operatorname{Re} \int_{\Gamma_h} \bar{u} T_1 u ds - \operatorname{Re} \int_{\Gamma_{-h}} \bar{u} T_2 u ds = -\operatorname{Re} \int_{\Omega_h} g \bar{u} dx \quad (3.6)$$

$$\operatorname{Im}(k_2^2) \int_{\Omega_{2,h}} |u|^2 dx + \operatorname{Im} \int_{\Gamma_h} \bar{u} T_1 u ds + \operatorname{Im} \int_{\Gamma_{-h}} \bar{u} T_2 u ds + \int_{\Gamma_1} \beta |u|^2 ds = \operatorname{Im} \int_{\Omega_h} g \bar{u} dx \quad (3.7)$$

By the standard elliptic regularity estimate [15] and since  $g \in L^2(\Omega_h)$ ,  $\Gamma \in C^2$  and  $\partial D \in C^2$ , we have  $u \in H_{\text{loc}}^2(\Omega_h)$ . For  $A > 0$ , let  $\varphi_A(\cdot) \in C_0^\infty(\mathbb{R})$  be a smooth cut-off function such that  $0 \leq \varphi_A(r) \leq 1$ ,  $\varphi_A(r) = 1$  if  $r \leq A$ ,  $\varphi_A = 0$  if  $r \geq A + 1$ , and  $\|\varphi_A'\|_{L^\infty(\mathbb{R})} < \infty$ . Applying the Green's theorem to  $u$  and  $\varphi_A(|x_1|)(x_2 + h)\partial\bar{u}/\partial x_2$  in  $\Omega_{1,h}(A)$  and  $\Omega_{2,h}(A)$  and letting  $A \rightarrow \infty$ , we have

$$\begin{aligned} 2h \int_{\Gamma_h} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 + k_1^2 |u|^2 \right) ds + \int_{\Gamma} (x_2 + h) \left( (|\nabla u^+|^2 - k_1^2 |u^+|^2) \nu_2 - 2 \operatorname{Re} \left( \frac{\partial u^+}{\partial \nu} \frac{\partial \bar{u}^+}{\partial x_2} \right) \right) ds \\ + \int_{\Omega_{1,h}} \left( |\nabla u|^2 - 2 \left| \frac{\partial u}{\partial x_2} \right|^2 - k_1^2 |u|^2 \right) dx = 2 \operatorname{Re} \int_{\Omega_{1,h}} (x_2 + h) g \frac{\partial \bar{u}}{\partial x_2} dx \end{aligned} \quad (3.8)$$

$$\begin{aligned}
& \int_{\Omega_{2,h}} \left( |\nabla u|^2 - 2 \left| \frac{\partial u}{\partial x_2} \right|^2 - \operatorname{Re}(k_2^2) |u|^2 \right) dx - \int_{\Gamma} \left( (|\nabla u^-|^2 - \operatorname{Re}(k_2^2) |u^-|^2) \nu_2 - 2 \operatorname{Re} \left( \frac{\partial u^-}{\partial \nu} \frac{\partial \bar{u}}{\partial x_2} \right) \right) ds \\
& + \int_{\partial D} (x_2 + h) \left( (|\nabla u^-|^2 - \operatorname{Re}(k_2^2) |u^-|^2) \nu_2 - 2 \operatorname{Re} \left( \frac{\partial u^-}{\partial \nu} \frac{\partial \bar{u}}{\partial x_2} \right) \right) ds - \operatorname{Im}(k_2^2) \operatorname{Im} \int_{\Omega_{2,h}} (x_2 + h) u_2 \frac{\partial \bar{u}}{\partial x_2} dx \\
& = 2 \operatorname{Re} \int_{\Omega_{2,h}} g(x_2 + h) \frac{\partial \bar{u}}{\partial x_2} dx
\end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9) together gives the Rellich identity

$$\begin{aligned}
& (k_1^2 - \operatorname{Re}(k_2^2)) \int_{\Gamma} (x_2 + h) |u|^2 \nu_2 ds + 2 \int_{\Omega_h} \left| \frac{\partial u}{\partial x_2} \right|^2 dx = \\
& 2h \int_{\Gamma_h} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 + k_1^2 |u|^2 \right) ds + \int_{\Omega_h} (|\nabla u|^2 - \operatorname{Re}(k^2) |u|^2) dx \\
& + \int_{\partial D} (x_2 + h) \left( (|\nabla u|^2 - \operatorname{Re}(k_2^2) |u|^2) \nu_2 - 2 \operatorname{Re} \left( \frac{\partial u}{\partial \nu} \frac{\partial \bar{u}}{\partial x_2} \right) \right) ds - \operatorname{Im}(k_2^2) \operatorname{Im} \int_{\Omega_{2,h}} (x_2 + h) u \frac{\partial \bar{u}}{\partial x_2} dx \\
& - 2 \operatorname{Re} \int_{\Omega_h} (x_2 + h) g \frac{\partial \bar{u}}{\partial x_2} dx
\end{aligned} \tag{3.10}$$

Now let  $D'$  be a bounded domain with  $\partial D' \in C^2$  such that  $D \subset D' \subset \Omega_{2,h} \cup \overline{D}$ . By the global elliptic global regularity estimate we get

$$\|u\|_{H^2(D' \setminus D)}^2 \leq C(\|u\|_{L^2(D' \setminus D)}^2 + \|g\|_{L^2(D' \setminus D)}^2) \tag{3.11}$$

Thus, by the trace theorem it follows that

$$\begin{aligned}
& \int_{\partial D} (x_2 + h) \left( (|\nabla u|^2 - \operatorname{Re}(k_2^2) |u|^2) \nu_2 - 2 \operatorname{Re} \left( \frac{\partial u}{\partial \nu} \frac{\partial \bar{u}}{\partial x_2} \right) \right) ds \leq \\
& C \|u\|_{H^2(D' \setminus D)}^2 \leq C(\|u\|_{L^2(D' \setminus D)}^2 + \|g\|_{L^2(D' \setminus D)}^2) \leq C(\|u\|_{L^2(\Omega_{2,h})}^2 + \|g\|_{L^2(\Omega_h)}^2)
\end{aligned} \tag{3.12}$$

From (2.9) and (2.10), and by using (3.6), (3.7) and the fact that  $\beta(x) \geq 0, x \in \Gamma_1$ , we have

$$\int_{\Omega_h} (|\nabla u|^2 - \operatorname{Re}(k^2) |u|^2) dx \leq -\operatorname{Re} \int_{\Omega_h} g \bar{u} dx, \tag{3.13}$$

$$\operatorname{Im}(k_2^2) \int_{\Omega_{2,h}} |u|^2 dx + \operatorname{Im} \int_{\Gamma_h} \bar{u} T_1 u ds \leq \operatorname{Im} \int_{\Omega_h} g \bar{u} dx. \tag{3.14}$$

By (2.12) and the definition of  $T_1$ , we have

$$\int_{\Gamma_h} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 + k_1^2 |u|^2 \right) ds \leq 2k_1 \operatorname{Im} \int_{\Gamma_h} \bar{u} \frac{\partial u}{\partial x_2} ds = 2k_1 \operatorname{Im} \int_{\Gamma_h} \bar{u} T_1 u ds \quad (3.15)$$

$$\leq 2k_1 \operatorname{Im} \int_{\Omega_h} g \bar{u} dx. \quad (3.16)$$

Thus, combining (3.10) and (3.12)-(3.15), and by Cauchy-Schwarz inequality, we get

$$\begin{aligned} & (k_1^2 - \operatorname{Re}(k_2^2)) \int_{\Gamma} (x_2 + h) |u|^2 \nu_2 ds + \int_{\Omega_h} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \leq \\ & C \left( \|g\|_{L^2(\Omega_h)} \|u\|_{H^1(\Omega_h)} + \|u\|_{L^2(\Omega_{2,h})} \left\| \frac{\partial u}{\partial x_2} \right\|_{L^2(\Omega_{2,h})} + \|g\|_{L^2(\Omega_h)}^2 \right). \end{aligned} \quad (3.17)$$

Applying Young's inequality to the second term on the right hand side of (3.17) and using (3.14), we obtain

$$\begin{aligned} & (k_1^2 - \operatorname{Re}(k_2^2)) \int_{\Gamma} (x_2 + h) |u|^2 \nu_2 ds + \int_{\Omega_h} \left| \frac{\partial u}{\partial x_2} \right|^2 dx \leq \\ & C(\|g\|_{L^2(\Omega_h)} \|u\|_{H^1(\Omega_h)} + \|g\|_{L^2(\Omega_h)}^2). \end{aligned} \quad (3.18)$$

On the other hand, (3.13) implies that

$$\|u\|_{H^1(\Omega_h)}^2 \leq (1 + \|k(x)\|_{L^\infty(\Omega_h)}) \|u\|_{L^2(\Omega_h)}^2 + \|g\|_{L^2(\Omega_h)} \|u\|_{L^2(\Omega_h)} \quad (3.19)$$

Further, we have the following inequality

$$\|u\|_{L^2(\Omega_{1,h})}^2 \leq 2h \|u\|_{L^2(\Gamma)}^2 + 2h^2 \left\| \frac{\partial u}{\partial x_2} \right\|_{L^2(\Omega_{1,h})}^2. \quad (3.20)$$

which can be proved similarly as in the proof of Lemma 4.3 in [22].

Combining (3.14), (3.18)-(3.20) and the fact that  $\|u\|_{L^2(\Omega_h)}^2 = \|u\|_{L^2(\Omega_{1,h})}^2 + \|u\|_{L^2(\Omega_{2,h})}^2$ , it follows that

$$\|u\|_{H^1(\Omega_h)}^2 \leq C(\|g\|_{L^2(\Omega_h)} \|u\|_{H^1(\Omega_h)} + \|g\|_{L^2(\Omega_h)}^2). \quad (3.21)$$

Applying Young's inequality to (3.21) yields

$$\|u\|_{H^1(\Omega_h)}^2 \leq C \|g\|_{L^2(\Omega_h)}^2. \quad (3.22)$$

The proof is complete.  $\square$

**Theorem 3.4.** *For every  $\mathcal{G} \in V_h^*$ , the variational problem (3.1) has a unique solution  $u \in V_h$  and*

$$\|u\|_{V_h} \leq C \|G\|_{V_h^*}. \quad (3.23)$$

*In particular, the variational problem (2.14) or the problem (BVP) is well-posed, and the solution satisfies the estimate*

$$\|u\|_{V_h} \leq C \|g\|_{L^2(\mathbb{R}^2)}. \quad (3.24)$$

*Proof.* From the boundedness of the sesquilinear form  $a(\cdot, \cdot)$  and the a priori estimate (3.5), and by arguing similarly as in the proof of Lemma 4.5 in [9], we can obtain a the priori estimate (3.3). Then by Lemma 3.2, the sesquilinear form  $a(\cdot, \cdot)$  satisfies the following inf-sup condition

$$\inf_{0 \neq u \in V_h} \sup_{0 \neq v \in V_h} \frac{|a(u, v)|}{\|u\|_{V_h} \|v\|_{V_h}} > 0.$$

Further, since  $a(u, v) = a(\overline{v}, \overline{u})$ , the following transposed inf-sup condition is also satisfied:

$$\sup_{0 \neq u \in V_h} \frac{|a(u, v)|}{\|u\|_{V_h}} > 0 \quad \text{for all } v \in V_h$$

Finally, by Theorem 3.1, we obtain the existence and uniqueness of solution of the variational problem (3.1) with the estimate (3.23). In particular, the estimate (3.24) also holds for the problem **(BVP)**, since, in this case,  $\mathcal{G}(v) = - \int_{\Omega_h} g \overline{v} dx$  with local source  $g \in L^2(\mathbb{R}^2)$  compactly supported in  $\Omega_h$  and  $\|\mathcal{G}\|_{V_h^*} \leq \|g\|_{L^2(\mathbb{R}^2)}$ . The proof is thus finished.  $\square$

### 3.2 Case 2: $0 < k_2^2 < k_1^2$

In this subsection, we consider the more challenging case with  $k_2^2 > 0$ . In this case, the integrals on  $\partial D$  in the Rellich identity (3.10) and  $\|u\|_{L^2(\Omega_{2,h})}$  can not be bounded by (3.14), so the a priori estimate (3.3) can not be established. However, it is seen from (3.12) that the integrals on  $\partial D$  in (3.10) can be bounded locally by  $C(\|u\|_{L^2(D' \setminus D)}^2 + \|g\|_{L^2(D' \setminus D)}^2)$ . This fact motivates us to find bounds for  $\|u\|_{L^2(D' \setminus D)}^2$  instead of  $\|u\|_{L^2(\Omega_{2,h})}^2$  with  $D \subset D' \subset \Omega_{2,h} \cup \overline{D}$ . Thus we first consider the variational problem (3.1) with the wave number defined by

$$k_\alpha^2(x) := \begin{cases} k_1^2, & x \in \Omega_1 \\ k_2^2 + i\alpha, & x \in D' \setminus D \\ k_2^2, & x \in \Omega_2 \setminus D' \end{cases}$$

with  $\alpha > 0$ .

**Theorem 3.5.** *The operator equation (2.16) with the wave number  $k = k_\alpha$  has a unique solution  $u \in V_h$ , that is,  $\mathcal{A}_{k_\alpha}^{-1} : V_h^* \rightarrow V_h$  is bounded.*

*Proof.* By the arguments used in the last subsection, it is sufficient to prove that the a priori estimate in Lemma 3.3 holds with the wave number in the sesquilinear form  $a(\cdot, \cdot)$  replaced by  $k_\alpha$ . One can then obtain the following Rellich identity

$$\begin{aligned} & (k_1^2 - k_2^2) \int_{\Gamma} (x_2 + h) |u|^2 \nu_2 ds + 2 \int_{\Omega_h} \left| \frac{\partial u}{\partial x_2} \right|^2 dx = \\ & 2h \int_{\Gamma_h} \left( \left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 + k_1^2 |u|^2 \right) ds + \int_{\Omega_h} (|\nabla u|^2 - k^2(x) |u|^2) dx \\ & + \int_{\partial D} (x_2 + h) \left( (|\nabla u|^2 - \operatorname{Re}(k_2^2) |u|^2) \nu_2 - 2 \operatorname{Re} \left( \frac{\partial u}{\partial \nu} \frac{\partial \overline{u}}{\partial x_2} \right) \right) ds \\ & - \alpha \operatorname{Im} \int_{D' \setminus D} (x_2 + h) u \frac{\partial \overline{u}}{\partial x_2} dx - 2 \operatorname{Re} \int_{\Omega_h} (x_2 + h) g \frac{\partial \overline{u}}{\partial x_2} dx \end{aligned} \tag{3.25}$$

and that (3.14) is replaced by

$$\alpha \int_{D' \setminus D} |u|^2 dx + \operatorname{Im} \left( \int_{\Gamma_h} \bar{u} T_1 u ds \right) \leq \operatorname{Im} \left( \int_{\Omega_h} g \bar{u} \right). \quad (3.26)$$

It is easy to see that (3.18) and (3.19) still hold. However, to bound  $\|u\|_{L^2(\Omega_h)}$ , we first extend  $u$  to  $\tilde{u}$  in  $\Omega_h \cup \overline{D}$  by defining  $\tilde{u} := u$  in  $\Omega_h$  and  $\tilde{u} := v$  in  $\overline{D}$ , where  $v$  is the solution to the following Dirichlet problem

$$\begin{aligned} \Delta v &= 0 \text{ in } D \\ v &= u|_{\partial D} \text{ on } \partial D \end{aligned}$$

Since  $u|_{\partial D} \in H^{\frac{1}{2}}(\partial D)$ , and by (3.11),  $\|v\|_{H^1(D)}^2 \leq C\|u\|_{H^{\frac{1}{2}}(\partial D)}^2 \leq C\|u\|_{H^1(D' \setminus D)}^2 \leq C(\|u\|_{L^2(D' \setminus D)}^2 + \|g\|_{L^2(D' \setminus D)}^2)$ . It is clear that  $\tilde{u} \in H^1(\Omega_h \cup \overline{D})$  and

$$\|\tilde{u}\|_{L^2(\Omega_h \cup \overline{D})}^2 \leq 4h\|\tilde{u}\|_{L^2(\Gamma)}^2 + 4h^2\left\|\frac{\partial \tilde{u}}{\partial x_2}\right\|_{L^2(\Omega_h \cup \overline{D})}^2. \quad (3.27)$$

Thus we have

$$\|u\|_{L^2(\Omega_h)}^2 \leq C(\|u\|_{L^2(\Gamma)}^2 + \left\|\frac{\partial u}{\partial x_2}\right\|_{L^2(\Omega_h)}^2 + \|u\|_{L^2(D' \setminus D)}^2 + \|g\|_{L^2(D' \setminus D)}^2) \quad (3.28)$$

From (3.12), (3.15), (3.19), (3.25), (3.26) and (3.28), it follows that

$$\|u\|_{V_h} \leq C\|g\|_{L^2(\Omega_h)}$$

The proof is finished.  $\square$

Now we come back to the analysis of the variational formulation with real wave numbers.

**Theorem 3.6.** *For the wave number  $k$  satisfying  $0 < k_2^2 < k_1^2$ ,  $\mathcal{A}_k : V_h \rightarrow V_h^*$  is a Fredholm operator with index of zero.*

*Proof.* Define the restriction operator  $\mathcal{P} : V_h \rightarrow V_h^*$  such that  $\mathcal{P}u = u|_{D' \setminus D}$  for  $u \in V_h$ , then  $\mathcal{P}$  is compact. This can be seen by the facts that the embedding  $V_h \rightarrow H^1(D' \setminus D)$  is bounded, the embedding  $H^1(D' \setminus D) \rightarrow L^2(D' \setminus D)$  is compact and the embedding  $L^2(D' \setminus D) \rightarrow V_h^*$  is bounded. Then, by the definition of  $\mathcal{A}_k$  and  $k_\alpha$ , we have  $\mathcal{A}_k = \mathcal{A}_{k_\alpha} - i\alpha\mathcal{P}$ . Thus  $\mathcal{A}_k u = \mathcal{G}$  can be rewritten as  $(\mathcal{A}_{k_\alpha} - i\alpha\mathcal{P})u = \mathcal{G}$ , where  $\mathcal{A}_{k_\alpha}$  is an isomorphism and  $\mathcal{P}$  is compact from  $V_h$  to  $V_h^*$ . Hence, it follows that  $\mathcal{A}_k$  is a Fredholm operator of index zero.  $\square$

**Corollary 3.7.** *Let the wave number  $k$  satisfy the condition in Theorem 3.6. If  $m(\Gamma_1) \neq 0, \beta > 0$  on  $\Gamma_1$ , where  $m(\Gamma_1)$  denotes the measure of  $\Gamma_1$  on the boundary  $\partial D$ , then there exists a unique solution to (2.16). In particular, the variational problem (2.14) or the problem (BVP) is well-posed with the solution satisfying the estimate (3.24).*

*Proof.* From Theorem 3.6, the existence follows from the uniqueness. It is sufficient to prove that if  $u \in V_h$  satisfying (2.16) with  $\mathcal{G} = 0$  then  $u$  vanishes in  $\Omega_h$ . Let  $v = u$  in (3.1) and take the real part of the equation (3.1). One obtains that

$$\int_{\Gamma_1} \beta |u| ds = 0$$

Since  $\beta > 0$  on  $\Gamma_1$ , we have  $u = 0$  on  $\Gamma_1$ , which together with the boundary condition (2.4) implies  $\frac{\partial u}{\partial \nu} = 0$  on  $\Gamma_1$ . By Holmgren's uniqueness theorem,  $u$  vanishes in  $\Omega_h$ . The well-posedness of the variational problem (2.16) or the problem **(BVP)** follows by the same argument used in Theorem 3.4.  $\square$

**Remark 3.8.** *In Corollary 3.7, the direct scattering problem is well-posed if the buried obstacle  $D$  is partially coated with a non-absorbing material. However, similar results can not be generalized to the cases with other boundary conditions (e.g., Dirichlet or Neumann boundary condition or mixed of Dirichlet and Neumann condition) on the obstacle since the uniqueness of solutions is not clear in these cases.*

In the end of this section, we give the following Corollary which will be used in the proof of Theorem 5.4.

**Corollary 3.9.** *Assume that the wave numbers belongs to case 1 or in case 2 and that the boundary of buried obstacle being partly dielectric, given  $f \in H^{-1}(\mathbb{R}^2)$  and a cut off function  $\chi$  with compact support  $K \subset \Omega_h$ , then there exists exactly one solution  $u \in V_h$  to **(BVP)** with  $g$  replaced by  $\chi f$ . And the solution  $u$  admits the estimate*

$$\|u\|_{V_h} \leq C \|f\|_{H^{-1}(\mathbb{R}^2)}$$

where the constant  $C > 0$  is independent of  $f$ .

*Proof.* We first claim that  $\chi f \in (H^1(K))^*$ . In fact, for  $\varphi \in C^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$

$$|\langle \chi f, \varphi \rangle| = |\langle f, \chi \varphi \rangle| \leq C \|f\|_{H^{-1}(\mathbb{R}^2)} \|\varphi\|_{H^1(K)}$$

where  $\langle \cdot, \cdot \rangle$  is the dual pair between  $H^{-1}(\mathbb{R}^2)$  and  $H^1(\mathbb{R}^2)$  and  $C$  is independent of  $f$ . By the argument of density, we obtain  $\|\chi f\|_{(H^1(K))^*} \leq C \|f\|_{H^{-1}(\mathbb{R}^2)}$ . Meanwhile, noting that  $K \subset \Omega_h$ , it is clear that  $V_h \subset H^1(\Omega_h) \subset H^1(K)$ . Therefore  $\chi f \in (H^1(K))^* \subset V_h^*$  and  $\|\chi f\|_{V_h^*} \leq C \|\chi f\|_{(H^1(K))^*}$ . Thus one have the following variational problem

$$a(u, v) = \langle \chi f, v \rangle_{V_h}$$

It is well-posed by Theorem 3.4 and Corollary 3.7, the proof is finished.  $\square$

**Remark 3.10.** *All the results in this section also hold under the condition that  $k_1^2 < \text{Re } k_2^2$ . In fact, Lemma 3.3 holds on noticing that applying Green's first theorem to  $u$  and  $\varphi_A(r)(x_2 - h)\partial\bar{u}/\partial x_2$  leads to a Rellich identity similar to (3.25). Then other results follow naturally after repeating the argument in this section again. And they also have the natural generalization in the cases of higher dimensions.*

**Remark 3.11.** *It is known from the proof of Lemma 3.3 and Theorem 3.5 that, for  $D = \emptyset$ , the direct scattering problem is well-posed if  $\text{Im}(k_2^2) \geq 0$  and either  $0 < k_1^2 < \text{Re}(k_2^2)$  or  $k_1^2 > \text{Re } k_2^2 > 0$ .*

In the remaining part of this paper, we always assume that one of the following conditions is satisfied, under which the direct scattering problem is well-posed:

- (i)  $0 < k_1^2 < \text{Re } k_2^2$  or  $k_1^2 > \text{Re } k_2^2 > 0$ ,  $\text{Im } k_2^2 > 0$ , and any boundary condition on the obstacle.
- (ii)  $0 < k_1^2 < k_2^2$  or  $k_1^2 > k_2^2 > 0$  and part of the obstacle is partly coated.

## 4 The scattering problem with incident point sources and hyper singular point sources

In this section, we study the well-posedness of the scattering problem corresponding to incident point source waves (PSWs) and hyper-singular point source waves (HSPSWs). The first case corresponds to the problem **(BVP)** with  $g$  being a Dirac delta function, saying  $\delta(x - z)$ ,  $z \in \mathbb{R}^2 \setminus \{\overline{D} \cup \Gamma\}$ , while for the second case,  $g = \delta'_1(x - z)$  where  $\delta'_1(x)$  stands for the derivative of  $\delta(x)$  with respect to  $x_1$  in the distributional sense. Obviously, Theorem 3.4 can not be applied directly since the distributions  $\delta(x - z)$  and  $\delta'_1(x - z)$  do not belong into  $V_h^*$ . However, we shall see shortly that both cases can be modified into the case that one can deal with by Theorem 3.4 and Corollary 3.7 and we only consider the case when the point source lies upon the rough interface.

Let  $\Phi_k(x; z) := i/4H_0^{(1)}(k|x - z|)$ ,  $x, z \in \mathbb{R}^2$ ,  $x \neq z$  denote the fundamental solution of the Hemholtz operator  $\Delta + k^2$  with  $H_0^{(1)}$  the Hankel function of the first kind of order zero. For  $z = (z_1, z_2) \in U_0^+$ , define  $z' = (z_1, -z_2)$ . Then  $G_k(x; z) = \Phi_k(x; z) - \Phi_k(x; z')$  is the Dirichlet Green's function for the Hemholtz operator  $\Delta + k^2$  in  $U_0^+$ . By the asymptotic property of the Hankel function for small and large arguments,  $G_k$  satisfies the following inequalities:

$$\begin{aligned} |G_k(x; z)|, |\nabla_x G_k(x; z)|, |\nabla_z G_k(x; z)| &\leq C \frac{(1 + |x_2|)(1 + |z_2|)}{|x - z|^{3/2}} \quad \text{for } x, z \in U_0^+ \text{ with } |x - z| \geq 1, \\ |G_k(x; z)| &\leq C(1 + |\log|x - z||) \quad \text{for } x, z \in U_0^+ \text{ with } 0 < |x - z| \leq 1, \\ |\nabla_x G_k(x; z)|, |\nabla_z G_k(x; z)| &\leq \frac{C}{|x - z|} \quad \text{for } x, z \in U_0^+ \text{ with } 0 < |x - z| \leq 1, \end{aligned} \tag{4.1}$$

where  $C$  is a positive constant depending only on  $k$ .

It is easy to verify that  $(\Delta + k^2)(\partial\Phi_{k_1}(x)/\partial x_1) = \delta'_1(x)$  in the distributional sense. Therefore,  $\partial\Phi_k(x)/\partial x_1$ ,  $x \neq 0$ , is the HSPSW positioned at the origin. Since  $G_{k_1}(\cdot; z)$  and  $G'_{k_1}(\cdot; z) := \partial G_{k_1}(\cdot; z)/\partial x_1$  both belong to  $L_{\text{loc}}^1(\mathbb{R}^2)$  and  $H^1(\Omega_h \setminus B_\varepsilon(z))$ ,  $\varepsilon > 0$ , for convenience, we may use  $G_{k_1}(x; z)$  (or  $G'_{k_1}(x; z)$ ) to denote the incident PSW (or incident HSPSW). Consider the incident field  $u^i \in \{G_{k_1}(\cdot; z), G'_{k_1}(\cdot; z)\}$ . We write the total field  $u^t = u^i + u$  in  $\Omega_1$  and  $u^t = u$  in  $\Omega_2$  with  $u$  being the transmitted field. Note that the total field corresponding to PSW and HSPSW does not belong to  $V_h$  because of the singularity of the incident field. However, for a source positioned at  $z \in \Omega_1$  and  $0 < \delta_0 < \text{dist}(z, \Gamma)$ , it is expected to find the solution in the space  $\widetilde{V}_h := \{u | u \in H^1(\Omega_h \setminus B_{\delta_0}(z)), u|_{\Gamma_2} = 0\}$  with the norm  $\|v\|_{\widetilde{V}_h} := \|v\|_{H^1(\Omega_h \setminus B_{\delta_0}(z))}$ .

**The scattering problem (SP):** For  $z \in \Omega_1$  and  $u^i \in \{G_{k_1}(\cdot; z), G'_{k_1}(\cdot; z)\}$ , find  $u^t(x; z) \in \widetilde{V}_h$ , such that

$$\begin{aligned} u^t(\cdot; z) &= u(\cdot; z) + u^i(\cdot; z) \quad \text{in } \Omega_1 \\ u^t(\cdot; z) &= u(\cdot; z) \quad \text{in } \Omega_2 \setminus D \end{aligned}$$

where  $u(\cdot; z)$  satisfies

$$\begin{aligned} \Delta u(\cdot; z) + k^2 u(\cdot; z) &= 0 \quad \text{in } \mathbb{R}^2 \setminus (\overline{D} \cup \Gamma) \\ u^+(\cdot; z) - u^-(\cdot; z) &= -u^i \quad \text{on } \Gamma, \\ \frac{\partial u^+}{\partial \nu}(\cdot; z) - \frac{\partial u^-}{\partial \nu}(\cdot; z) &= -\frac{\partial u^i}{\partial \nu} \quad \text{on } \Gamma, \end{aligned}$$

and the boundary condition (2.4) and the radiation conditions (2.5) and (2.6).

We study the existence and uniqueness of solutions to the scattering problem **(SP)** by replacing the incident wave with a non-singular one. This technique has been used in [8]. choose  $\delta > 0$  such that  $\text{dist}(z, \Gamma) > \delta$  and define a new incident wave by

$$\tilde{u}^i(x) = \begin{cases} u^i(x), & x \notin B_\delta(z) \\ A + BJ_0(x), & \text{otherwise} \end{cases} \quad (4.2)$$

where the constants  $A$  and  $B$  are chosen to ensure that  $\tilde{u}^i \in C^1(\Omega_{1,h})$ . Then  $\tilde{u}^i \in H_{\text{loc}}^2(\Omega_{1,h})$  and  $(\Delta + k_1^2)\tilde{u}^i = \tilde{g}$ , where  $\tilde{g}(x) := Ak_1^2$  for  $x \in B_\delta(y)$ ,  $\tilde{g}(x) := 0$  otherwise. Since  $u^i = \tilde{u}^i$  outside  $B_\delta(z)$  and  $B_\delta(z) \cap \Gamma = \emptyset$ ,  $u^i$  and  $\tilde{u}^i$  have the same boundary value and normal derivative on  $\Gamma$ . Obviously, the substitution of  $u^i$  by  $\tilde{u}^i$  does not change the scattered field, so we can reformulate the scattering problem **(SP)** by finding  $\tilde{u}^t(x) = \tilde{u}^i(x) + u(x)$  for  $x \in \overline{\Omega}_{1,h}$ ,  $\tilde{u}^t = u(x)$  for  $x \in \overline{\Omega}_{2,h}$  such that  $\tilde{u}^t$  solves **(BVP)** with  $g := \tilde{g} \in L^2(\mathbb{R}^2)$ . By Theorem 3.4 and Corollary 3.7, there is a unique solution  $\tilde{u}^t \in V_h$  to the scattering problem **(SP)**. Then we obtain the scattered field  $u(x) = \tilde{u}^t(x) - \tilde{u}^i(x)$  for  $x \in \overline{\Omega}_{1,h}$ ,  $u(x) = \tilde{u}^t(x)$  for  $x \in \overline{\Omega}_{2,h}$ . It is clear that  $u|_{\Omega_{j,h}} \in H^1(\Omega_{j,h})$ . Since  $G_{k_1}(x; z)$  and  $G_{k_1}(x; z)$  satisfy the inequality (4.1), we have  $u^t(\cdot; z) \in \widetilde{V}_h$ .

For  $z \in \Omega_1$  and  $\delta > 0$ , the solution to the scattering problem **(SP)** satisfies that  $u^t(\cdot; z) \in \widetilde{V}_h$ . Denote by  $K$  a compact set in  $\Omega_h \setminus B_\delta(z)$ , it is clear that  $\|u^t(\cdot; z)\|_{H^1(K)}$  is bounded. In fact, it can be shown that as  $z$  approaches  $\Gamma$ ,  $\|u^t(\cdot; z)\|_{H^1(K)}$  is bounded uniformly. This is shown in the following theorem, which will be one of the key ingredients in proving the uniqueness for the inverse scattering problem.

**Theorem 4.1.** *For the fixed  $z_0 \in \mathbb{R}^2 \setminus D$ ,  $\delta > 0$ ,  $z \in \overline{B_\delta(z_0)} \setminus \Gamma$  with  $\overline{B_\delta(z_0)} \cap \overline{D} = \emptyset$  and the compact set  $K \subset \Omega_h \setminus B_\delta(z_0)$ , assume that  $d = \text{dist}(K, \overline{B_\delta(z_0)}) > 0$ . Then the total field  $u^t(\cdot; z)$  of the scattering problem **(SP)** satisfies the estimate*

$$\|u^t(\cdot; z)\|_{H^1(K)} \leq C, \quad (4.3)$$

where the constant  $C > 0$  depends on  $k_1, k_2, \delta, d$  but is independent of  $z$ .

*Proof.* We only consider the case that  $z_0 \in \overline{\Omega}_1$ ,  $z \in \Omega_1$  and the incident field  $u^i$  is a HSPSW. The other cases can be proved similarly. For the case that the incident field being a PSW, see Remark 4.2.

Take a smooth cut-off function  $\chi(x)$  such that  $\chi(x) = 1$  for  $x \in B_\delta(z_0)$ ,  $\chi(x) = 0$  for  $x \in B_{\delta+d/2}^c(z_0)$ , and  $0 < \chi(x) < 1$ . Then the total field can be written as  $u^t(x) = \chi G'_{k_1}(x; z) + V(x; z)$ . It is clear that  $V(\cdot; z)$  satisfies

$$(\Delta + k^2)V(x; z) = \tilde{g}_z(x)$$

where  $\tilde{g}$  is defined by

$$\tilde{g}_z(x) = \begin{cases} -\Delta \chi G'_{k_1}(x; z) - 2\nabla \chi \cdot \nabla G'_{k_1}(x; z) & \text{in } \Omega_1, \\ -\Delta \chi G'_{k_1}(x; z) - 2\nabla \chi \cdot \nabla G'_{k_1}(x; z) + \chi(k_1^2 - k_2^2)G'_{k_1}(x; z) & \text{in } \Omega_2, \end{cases}$$

and by the definition of  $\chi$ ,  $V(\cdot; z)$  satisfies the transmission condition (2.3), the boundary condition (2.4) and the radiation conditions (2.5)-(2.6).

We claim that  $\tilde{g}_z \in H^{-1}(\mathbb{R}^2)$  with a compact support. Since  $-\Delta \chi G'_{k_1}(\cdot; z) - 2\nabla \chi \cdot \nabla G'_{k_1}(\cdot; z) \in L^2(\mathbb{R}^2)$  supported in  $B_{\delta+d/2}(z_0)$ , we only need to prove that  $f_z \in H^{-1}(\mathbb{R}^2)$



with  $\tilde{f}_z := 0$  in  $\Omega_1$ ,  $\tilde{f}_z := \chi(\cdot)(k_1^2 - k_2^2)G'_{k_1}(\cdot; z)$  in  $\Omega_2$ . In fact, it is seen from (4.1) that  $G'_{k_1}(\cdot; z) \in L^p_{loc}(\mathbb{R}^2)$  with  $1 \leq p < 2$ . Then, taking  $1 < p < 2$ , we have  $\tilde{f}_z \in L^p(\mathbb{R}^2)$  with a compact support. By the standard Sobolev embedding theorem, we have that  $H^1_0(\mathbb{R}^2) \hookrightarrow L^q$  for all  $2 \leq q < \infty$ . Thus  $\tilde{f}_z \in L^p(\mathbb{R}^2) \subset H^{-1}(\mathbb{R}^2)$  for  $1 < p < 2$ . Furthermore, we have

$$\begin{aligned} \|\tilde{g}\|_{H^{-1}(\mathbb{R}^2)} &\leq \|\chi\|_{H^2(B_{\delta+d/2}(z_0) \setminus B_\delta(z_0))} \|G'_{k_1}\|_{H^1(B_{\delta+d/2}(z_0) \setminus B_\delta(z_0))} + \\ &|k_1^2 - k_2^2| \|\chi\|_{H^1(B_{\delta+d/2}(z_0))} \|G_{k_1}\|_{L^2(B_{\delta+d/2}(z_0))} \leq C(k_1, k_2, \delta, d) \end{aligned}$$

Since  $\tilde{g}_z$  is compactly supported in  $B_{\delta+d/2}(z_0)$ , then for another smooth cut-off function  $\tilde{\chi}$  such that  $\tilde{\chi} = 1$  in  $B_{\delta+d/2}(z_0)$  and  $\tilde{\chi} = 0$  outside  $B_{\delta+3d/4}(z_0)$ , we have  $\tilde{\chi}\tilde{g}_z = \tilde{g}_z$ . From Corollary 3.9, we see that  $\|V(x; z)\|_{\widehat{V}_h} \leq C(\delta, d)\|\tilde{g}\|_{H^{-1}(\mathbb{R}^2)} \leq C(k_1, k_2, \delta, d)$  for any  $z \in B_\delta(z_0) \cap \Omega_1$ . Since  $u^t(x) = \chi G'_{k_1}(x; z) + V(x; z)$ , and by the definition of  $\chi$ , (4.3) holds.  $\square$

**Remark 4.2.** (i) For **(SP)** due to PSW, by the same argument used in Theorem 4.1, we can conclude that  $u^t(\cdot; z) \in H^2(K)$  and  $\|u^t(\cdot; z)\|_{H^2(K)}$  is bounded uniformly with respect to  $z$ .

(ii) In the case that  $z \in B_\delta(z_0)$  with  $\text{dist}(B_\delta(z_0), \Gamma) = d > 0$ ,  $u(\cdot; z) \in H^1_{loc}(\Omega_j)$ ,  $j = 1, 2$ , and  $\|u(\cdot; z)\|_{H^1_{loc}(\Omega_j)}$  is bounded uniformly with respect to  $z$ .

Denote by  $u(\cdot; z)$  ( $u^t(\cdot; z)$ ) the scattered (total) field corresponding to an incident PSW with the source position  $z$  and by  $u'(\cdot; z)$  ( $u'^t(\cdot; z)$ ) the scattered (total) field corresponding to an incident HSPSW positioned at  $z$ . Define  $\widehat{V}_h := \{u|_{\Omega_{1,h}} \in H^1(\Omega_{1,h}), u|_{\Omega_{2,h}} \in H^1(\Omega_{2,h}), u|_{\Gamma_2} = 0\}$  with the norm  $\|u\|_{\widehat{V}_h} = \|u\|_{H^1(\Omega_{1,h})} + \|u\|_{H^1(\Omega_{2,h})}$ . The following Theorem gives an important relation between  $u(\cdot; z)$  and  $u'(\cdot; z)$ .

**Theorem 4.3.** For  $z \in \Omega_1$ , the limit

$$\frac{\partial u(\cdot; z)}{\partial z_1} := \lim_{\varepsilon \rightarrow 0} \frac{u(\cdot; z + \varepsilon \mathbf{e}_1) - u(\cdot; z)}{\varepsilon}$$

exists in  $\widehat{V}_h$ , where  $\mathbf{e}_1 = (1, 0)^T$ . Further,  $u'(\cdot; z) = -\frac{\partial u(\cdot; z)}{\partial z_1}$

*Proof.* It is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} v_\varepsilon := \lim_{\varepsilon \rightarrow 0} \left( \frac{u(\cdot; z + \varepsilon \mathbf{e}_1) - u(\cdot; z)}{\varepsilon} + u'(\cdot; z) \right) = 0$$

in  $\widehat{V}_h$ .

Noting that  $\partial G_{k_1,z}(x; z)/\partial x_1 = -\partial G_{k_1,z}(x; z)/\partial z_1$ ,  $x \in \Gamma$ , it is clear that  $v_\varepsilon$  is the scattering solution of the scattering problem **(SP)** with  $u^i = u^i_\varepsilon := (G_{k_1,z}(x; z + \varepsilon \mathbf{e}_1) - G_{k_1,z}(x; z))/\varepsilon - \partial G_{k_1,z}(x; z)/\partial z_1$ . Then the total field  $u^t_\varepsilon = v_\varepsilon + u^i_\varepsilon$  in  $\Omega_1$ ,  $u^t_\varepsilon = v_\varepsilon$  in  $\Omega_2$ . Since  $z \in \Omega_1$ , there exists a  $r > 0$  such that  $\overline{B_r(z)} \in \Omega_1$ . We may assume that  $|\varepsilon| < r/8$ . To study the asymptotic property of  $v_\varepsilon$  as  $\varepsilon \rightarrow 0$ , we first take a smooth cut-off function  $\chi(x)$ , such that  $\chi(x) = 1, x \in B_{r/4}(z)$ ,  $\chi(x) = 0, x \in B^c_{r/2}(z)$ . The total field can be written as  $u^t_\varepsilon = \chi u^i_\varepsilon + V_\varepsilon$  in  $\Omega_1$ ,  $u^t_\varepsilon = V_\varepsilon$  in  $\Omega_2$ . Then  $V_\varepsilon$  satisfies that

$$\Delta V_\varepsilon + k^2 V_\varepsilon = g_\varepsilon \quad \text{in } \mathbb{R}^2,$$

where  $g_\varepsilon = -(\Delta \chi u^i_\varepsilon + 2\nabla \chi \cdot \nabla u^i_\varepsilon)$ . Moreover,  $V_\varepsilon$  satisfies the transmission condition (2.3), the boundary condition (2.4) and the radiation conditions (2.5) and (2.6). By the definition of  $\chi$ ,

we see that  $g_\varepsilon$  is a smooth function compactly supported in  $B_{r/2}(z) \setminus B_{r/4}(z)$  and satisfies the estimate

$$\|g_\varepsilon\|_{L^2(\mathbb{R}^2)} = \|g\|_{L^2(B_{r/2}(z) \setminus B_{r/4}(z))} \leq C\|u_\varepsilon^i\|_{H^1(B_{r/2}(z) \setminus B_{r/4}(z))}$$

where the constant  $C$  is independent of  $\varepsilon$ . From Theorem 3.4, we have  $\|V_\varepsilon\|_{V_h} \leq C\|g_\varepsilon\|_{L^2(\mathbb{R}^2)}$ . Then  $v_\varepsilon = (1 - \chi)u_\varepsilon^{\text{in}} + V_\varepsilon$  in  $\Omega_1$ ,  $v_\varepsilon = V_\varepsilon$  in  $\Omega_2$ . Thus,

$$\begin{aligned} \|v_\varepsilon\|_{H^1(\Omega_{1,h})} + \|v_\varepsilon\|_{H^1(\Omega_{2,h})} &\leq C(\|u_\varepsilon^i\|_{H^1(\Omega_{1,h} \setminus B_{r/2}(z))} + \|V_\varepsilon\|_{V_h}) \leq \\ &C(\|u_\varepsilon^i\|_{H^1(\Omega_{1,h} \setminus B_{r/2}(z))} + \|g_\varepsilon\|_{L^2(\mathbb{R}^2)}) \leq C\|u_\varepsilon^i\|_{H^1(\Omega_{1,h} \setminus B_{r/4}(z))} \end{aligned} \quad (4.4)$$

By the asymptotic property of the point source and its derivatives at infinity we have  $u_\varepsilon^i \in H^1(\Omega_{1,h} \setminus B_{r/4}(z))$ . Then for a given  $\eta > 0$ , there exists a  $M > 0$  such that  $\|u_\varepsilon^i\|_{H^1(\mathbb{R}^2 \setminus \Omega_{1,h}(M))} \leq \eta/2$  for all  $\varepsilon$  satisfying  $|\varepsilon| < r/8$ . Since  $|\varepsilon| < r/8$ , by the interior elliptic regularity and the fact that  $u_\varepsilon^i \in H^1(\Omega_{1,h} \setminus B_{r/4}(z))$ , there exists a  $\delta > 0$  such that  $\|u_\varepsilon^i\|_{H^1(\Omega_{1,h}(M) \setminus B_{r/4}(z))} < \eta/2$  for  $|\varepsilon| < \delta$ . Therefore, for any  $\forall \eta > 0$  there exists a  $\delta > 0$  such that  $\|u_\varepsilon^i\|_{H^1(\Omega_{1,h} \setminus B_{r/4}(z))} < \eta$  for  $|\varepsilon| < \delta$ , so by (4.4)  $\|v_\varepsilon\|_{H^1(\Omega_{1,h})} + \|v_\varepsilon\|_{H^1(\Omega_{2,h})} \leq C\eta$ . This means that  $v_\varepsilon \rightarrow 0$  in  $\widehat{V}_h$  as  $\varepsilon \rightarrow 0$ . The proof is finished.  $\square$

**Remark 4.4.** *Theorem 4.3 also holds in higher dimensions. Nevertheless, Theorem 4.1 can only be proved in two and three dimensions up to now. In fact, for the case that  $n \geq 3$  in Theorem 4.1,  $\tilde{g}_z \in L^p(\mathbb{R}^n)$  with a compact support if  $1 \leq p < n/(n-1)$  and  $H_0^1(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{R}^n)$  or equivalently,  $L^{2n/(n+2)}(\mathbb{R}^n) \hookrightarrow H^{-1}(\mathbb{R}^2)$ . Then for a function  $f$  in  $L^p(\mathbb{R}^n)$  with  $1 \leq p \leq \infty$  with a compact support, by the Cauchy-Schwarz inequality we have that  $f \in L^q(\mathbb{R}^n)$  if  $1 \leq q \leq p$ . Thus we can conclude that  $\tilde{g}_z \in H^{-1}(\mathbb{R}^n)$  if there exists a  $p$  such that  $2n/(n+2) < p < n/(n-1)$  which implies that  $n < 4$ . By Corollary 3.9, Theorem 4.1 holds in two and three dimensions. However, it is not clear whether the same conclusion in dimensions  $n \geq 4$  holds since the solvability of the boundary value problem (BVP) with the right hand  $g \in L^p(1 \leq p < n/(n-1))$  is unknown yet.*

## 5 The inverse scattering problem

In this section, we consider the inverse problem of recovering the interface and the buried obstacle with its physical property simultaneously from the scattered field generated by PSWs. Suppose the scattered fields are generated by PSWs with source positions located on the line segment  $\Sigma_s \subset \Gamma_b$  and measured on another line segment  $\Sigma_r \subset \Gamma_c$ ; see Figure 1. Then the inverse scattering problem can be stated as follows.

**Inverse scattering problem (ISP)** Given the wave numbers  $k_j, j = 1, 2$ , and the scattered field  $u(x; z)$  for  $z \in \Sigma_s \subset \Gamma_b, x \in \Sigma_r \subset \Gamma_c$ , determine the rough interface  $\Gamma$ , the obstacle  $D$  and its physical property  $\mathcal{B}$ .

In our proof of the inverse scattering problem, the recovery of rough interface will be achieved by constructing a special transmission problem named **Interior Transmission Problem (ITP)**. In recent years, there have been great developments on the study of interior transmission problem and the associated transmission eigenvalues, see e.g. [3, 4, 39, 38, 37]. It is mainly because the knowledge of transmission eigenvalues is useful to retrieve information about scatters [40, 41]; the existence and discreteness of the eigenvalues are crucial in both scattering theory and its numerical computation [3, 42]. Recently, Yang, Zhang and Zhang [32] exploit

the interior transmission problem to study the inverse problem on bounded penetrable obstacle scattering problem. We will use the same idea to prove the uniqueness of the inverse rough surface scattering problem. To that end, we first briefly recall some background on the interior transmission problem. We define space

$$H_{\Delta}^1(\Omega) := \{w \in H^1(\Omega) \mid \Delta w \in L^2(\Omega)\}$$

equipped with the norm  $\|w\|_{H_{\Delta}^1(\Omega)}^2 = \|w\|_{H^1(\Omega)}^2 + \|\Delta w\|_{L^2(\Omega)}^2$ . It is clear that  $H_{\Delta}^1(\Omega)$  is a Hilbert space. Moreover, a function  $w \in H_{\Delta}^1(\Omega)$  has traces  $\gamma_0 w \in H^{\frac{1}{2}}(\partial\Omega)$  and  $\gamma_1 w \in H^{-\frac{1}{2}}(\partial\Omega)$ . In particular, we set  $H_0^2(\Omega) := \{w \in H_{\Delta}^1(\Omega) \mid \gamma_0 w = \gamma_1 w = 0\}$ . Let  $n(x)$  be a function of refraction index such that  $\text{Im}(n) \geq 0$ . Suppose that either  $1 + r_0 < \text{Re}(n) < \infty$  or  $0 < \text{Re}(n) < 1 - r_1$ , where  $r_0, r_1 > 0$ .

**Interior Transmission Problem (ITP):** Given  $(f_1, f_2) \in \{(\gamma_0 w, \gamma_1 w) \mid w \in H_{\Delta}^1(\Omega)\}$ , find  $U, V \in L^2(\Omega)$  such that  $U - V - w \in H_0^2(\Omega)$  and satisfying

$$\begin{aligned} \Delta V + k^2 V &= 0 \quad \text{in } \Omega, \\ \Delta U + k^2 n U &= 0 \quad \text{in } \Omega, \\ U - V &= f_1, \quad \frac{\partial U}{\partial \nu} - \frac{\partial V}{\partial \nu} = f_2 \quad \text{on } \partial\Omega \end{aligned}$$

We say  $k^2$  is an interior transmission eigenvalue of **(ITP)** if the homogenous problem has nonzero solution. An interior transmission problem is well-posed if  $k^2$  is not an interior transmission eigenvalue. In the following theorem, we collect some results about the wellposedness of **(ITP)** and properties of the interior transmission eigenvalues.

**Theorem 5.1.**

(i) Let  $\Omega$  be fixed. If  $\text{Im}(n) > 0$ , then **(ITP)** (ii) well-posed. If  $\text{Im}(n) = 0$ , then there exists an infinite set of transmission eigenvalues of **(ITP)** with  $+\infty$  as the only accumulation point. Moreover, if  $k^2$  is not an eigenvalue, then **(ITP)** has a unique solution  $(U, V) \in L^2(\Omega) \times L^2(\Omega)$  such that

$$\|U\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)} \leq C(\|f_1\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|f_2\|_{H^{-\frac{1}{2}}(\partial\Omega)}) \quad (5.1)$$

(ii) Let  $k > 0$  be fixed and assume that  $\text{Im}(n) = 0$ . If the diameter of the domain  $\Omega$  is small enough, then  $k^2$  can not be the interior transmission eigenvalue of **(ITP)**. Moreover, in such case the estimate (5.1) holds.

The proof of Theorem 5.1 (i) can be found in [3, 4]. Theorem 5.1 (ii) is proved recently in [32]. The following reciprocity relation about the total field (as well as the scattered field) induced by the point sources will also be useful.

**Theorem 5.2. (Reciprocity relation)**

For  $z_1, z_2 \in \mathbb{R}^2 \setminus \{\Gamma \cup \overline{D}\}$  and  $z_1 \neq z_2$ , the total field satisfies

$$u^t(z_1; z_2) = u^t(z_2; z_1)$$

*Proof.* We only consider the case  $z_1 \in \Omega_1, z_2 \in \Omega_2 \setminus \overline{D}$ , the other cases can be treated similarly.

For  $A > 0$  and  $h > \max(|z_1|, |z_2|)$ , take  $\varepsilon > 0$  such that  $B_\varepsilon(z_1) \subset \Omega_{1,h}(A)$  and  $B_\varepsilon(z_2) \subset \Omega_{2,h}(A)$ . Since  $u^\dagger(\cdot; z_j) \in H_{\text{loc}}^2(\Omega_{j,h}(A) \setminus B_\varepsilon(z_j))$ , we apply Green's first theorem to  $u^\dagger(x; z_1)$  and  $u^\dagger(x; z_2)$  in  $\Omega_{1,h}(A) \setminus B_\varepsilon(z_1)$  to get

$$\begin{aligned}
0 &= \int_{\Omega_{1,h}(A)} (u^\dagger(x; z_1) \Delta u^\dagger(x; z_2) - u^\dagger(x; z_2) \Delta u^\dagger(x; z_1)) \, dx = \\
&\int_{\Gamma_h(A)} \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial \nu}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right) \, ds + R_1(A) - \\
&\int_{\Gamma(A)} \left( u^\dagger(x; z_1) \left| \frac{\partial u^\dagger}{\partial \nu}(x; z_2) \right|^+ - u^\dagger(x; z_2) \left| \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right|^+ \right) \, ds - \\
&\int_{\partial B_\varepsilon(z_1)} \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial \nu}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right) \, ds
\end{aligned} \tag{5.2}$$

where

$$R_1(A) := \left[ \int_{\gamma_1(A)} - \int_{\gamma_1(-A)} \right] \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial x_2}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial x_2}(x; z_1) \right) \, ds$$

Similarly, applying Green's first theorem to  $u^\dagger(\cdot; z_1)$  and  $u^\dagger(\cdot; z_2)$  in  $\Omega_{2,h}(A) \setminus B_\varepsilon(z_2)$  gives

$$\begin{aligned}
0 &= \int_{\Omega_{2,h}(A)} (u^\dagger(x; z_1) \Delta u^\dagger(x; z_2) - u^\dagger(x; z_2) \Delta u^\dagger(x; z_1)) \, dx = \\
&\int_{\Gamma_{-h}(A)} \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial \nu}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right) \, ds + R_2(A) + \\
&\int_{\Gamma(A)} \left( u^\dagger(x; z_1) \left| \frac{\partial u^\dagger}{\partial \nu}(x; z_2) \right|^- - u^\dagger(x; z_2) \left| \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right|^- \right) \, ds - \\
&\int_{\partial B_\varepsilon(z_2)} \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial \nu}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right) \, ds - \\
&\int_{\partial D} \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial \nu}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial \nu}(x; z_1) \right) \, ds
\end{aligned} \tag{5.3}$$

where

$$R_2(A) := \left[ \int_{\gamma_2(A)} - \int_{\gamma_2(-A)} \right] \left( u^\dagger(x; z_1) \frac{\partial u^\dagger}{\partial x_2}(x; z_2) - u^\dagger(x; z_2) \frac{\partial u^\dagger}{\partial x_2}(x; z_1) \right) \, ds$$

Since  $u^\dagger(\cdot; z_1)$  and  $u^\dagger(\cdot; z_2)$  satisfy the Helmholtz equation in  $\Omega_{2,h}(A) \setminus B_\varepsilon(z_2)$ , the transmission

conditions on  $\Gamma$  and the boundary conditions on  $\partial D$ , (5.2) and (5.3) yields

$$\begin{aligned}
0 = & \int_{\Gamma_h(A)} \left( u^t(x; z_1) \frac{\partial u^t(x; z_2)}{\partial \nu(x)} - u^t(x; z_2) \frac{\partial u^t(x; z_1)}{\partial \nu(x)} \right) ds + \\
& \int_{\Gamma_{-h}(A)} \left( u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) \right) ds + \\
R_1(A) - & \int_{\partial B_\varepsilon(z_1)} \left( u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) \right) ds + \\
R_2(A) - & \int_{\partial B_\varepsilon(z_2)} \left( u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) \right) ds.
\end{aligned} \tag{5.4}$$

As  $\varepsilon \rightarrow 0$ , we have

$$\begin{aligned}
& \int_{\partial B_\varepsilon(z_1)} \left( u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) \right) ds = \\
& \int_{\partial B_\varepsilon(z_1)} \left( G_{k_1}(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial G_{k_1}}{\partial \nu}(x; z_1) \right) ds + \\
& \int_{\partial B_\varepsilon(z_1)} \left( u(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u}{\partial \nu}(x; z_1) \right) ds \rightarrow u^t(z_1; z_2)
\end{aligned}$$

where we have used Theorem 2.1 in [11]. Similarly,

$$\int_{\partial B_\varepsilon(z_2)} \left( u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) \right) ds \rightarrow -u^t(z_2; z_1) \quad \text{as } \varepsilon \rightarrow 0.$$

Since  $u^t(\cdot; z_j) \in \widetilde{V}_h$ ,  $j = 1, 2$  and satisfy the angular-spectrum representation (2.5) and (2.6), and by (2.11) in Lemma 2.1, we have

$$\begin{aligned}
& \int_{\Gamma_h(A)} u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) ds = \\
& \int_{\Gamma_{-h}(A)} u^t(x; z_1) \frac{\partial u^t}{\partial \nu}(x; z_2) - u^t(x; z_2) \frac{\partial u^t}{\partial \nu}(x; z_1) ds \rightarrow 0
\end{aligned}$$

as  $A \rightarrow +\infty$ . Noting that  $u^t(\cdot, z_i) \in \widetilde{V}_h$ , it follows from the asymptotic properties of  $G_{k_1}(\cdot, z_i)$   $i = 1, 2$  and their derivatives that  $R_j(A) \rightarrow 0$  as  $A \rightarrow +\infty$ ,  $j = 1, 2$ . Thus,  $u^t(z_2; z_1) = u^t(z_1; z_2)$  follows from (5.4) by letting  $\varepsilon \rightarrow 0$  and  $A \rightarrow +\infty$ . The proof is complete.  $\square$

**Remark 5.3.** By the symmetry of  $G_k(x, z)$ , we also have the reciprocity relation of the scattered field.

$$u(z_1; z_2) = u(z_2; z_1), \quad z_1, z_2 \in \Omega_1 \text{ or } \Omega_2 \setminus \overline{D} \text{ and } z_1 \neq z_2 \tag{5.5}$$

Suppose that  $\Gamma$  and  $\tilde{\Gamma}$  are two rough interfaces and that  $D$  and  $\tilde{D}$  are two impenetrable obstacles with the boundary physical property  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  respectively. Define  $\tilde{u}(\cdot; z)$  and  $\tilde{u}^t(\cdot; z)$  to be the scattered and total field due to the PSW and given by the scattering problem **(SP)** with  $\tilde{\Gamma}, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{D}, \tilde{\mathcal{B}}$ . The fields  $u', u^t, \tilde{u}', \tilde{u}^t$  due to the HSPSW can be defined accordingly.

We now have the uniqueness result of the inverse scattering problem.

**Theorem 5.4.** *If the scattered field  $u(x; z) = \tilde{u}(x; z)$  for all  $z \in \Sigma_s \subset \Gamma_b$  and  $x \in \Sigma_r \subset \Gamma_c$ , then  $\Gamma = \tilde{\Gamma}, D = \tilde{D}, \mathcal{B} = \tilde{\mathcal{B}}$ .*

*Proof. Step 1.* We prove that  $\Gamma = \tilde{\Gamma}$ .

Let  $\Omega$  be the unbounded connected component of  $\Omega_1 \cap \tilde{\Omega}_1$ . For  $z \in \Omega$ , we first claim that

$$u(x; z) = \tilde{u}(x; z) \quad \text{for all } x \in \overline{\Omega} \quad (5.6)$$

Since  $u(\cdot; z)$  and  $\tilde{u}(\cdot; z)$  are both analytic in  $\Omega$  and  $u(x; z) = \tilde{u}(x; z)$  for all  $x \in \Sigma_r$ , then  $u(x; z) = \tilde{u}(x; z), x \in \Gamma_c$ . From the uniqueness of the Dirichlet problem in  $U_c^+$ , we know that (5.6) holds for  $x \in U_c^+, z \in \Sigma_s$ . By the unique continuation principle, (5.6) also holds for  $x \in \Omega, z \in \Sigma_s$ . By the reciprocity relation (5.5), we have  $u(z; x) = u(x; z)$  for  $z$  on  $\Sigma_s, x \in \Omega$ . Repeating the argument above, we obtain  $u(x; z) = \tilde{u}(x; z)$  for all  $z \in \Omega, x \in \Omega$ . Since the scattered fields are continuous up to the boundary, (5.6) holds. By Theorem 4.3 and (5.6) we have

$$u'(x; z) = \tilde{u}'(x; z) \quad \text{for all } z \in \Omega, x \in \overline{\Omega}$$

Assume that  $\Gamma \neq \tilde{\Gamma}$ . Without loss of generality, we assume that there exists  $z^* \in \Gamma \setminus \tilde{\Gamma}$ . Define  $z_j := z^* + (\delta/j)\nu(z^*), j \in \mathbb{N}^+$ , with  $\delta > 0$  such that  $z_j \in B_\delta(z^*)$  where  $B_\delta(z^*)$  is a ball centred at  $z^*$  and with radius  $\delta$  satisfying  $\overline{B_{2\delta}(z^*)} \subset \tilde{\Omega}_1$ . Choose a small domain  $\Omega_0 \subset \Omega_2$  with  $C^2$ -boundary  $\partial\Omega_0$  such that  $B_{2\delta} \cap \Omega_2 \subset \Omega_0 \subset \overline{\Omega_0} \subset \tilde{\Omega}_1$  and let  $d := \text{dist}(\Omega_0, \tilde{\Gamma}) > 0$ . See the geometric setting in Figure 2. Define the scattered field  $u'_j(x) := u'(x; z_j), \tilde{u}'_j(x) := \tilde{u}'(x; z_j)$  and the total field  $u_j^t(x) := u^t(x; z_j), \tilde{u}_j^t(x) := \tilde{u}^t(x; z_j)$ . Also we set  $V_j = \tilde{u}_j^t|_{\Omega_0}$  and  $U_j = u'_j|_{\Omega_0}$  in  $\Omega_0$ . Then  $V_j$  and  $U_j$  satisfy **(ITP)** in  $\Omega_0$  with boundary data  $f_{1,j} := (u'_j - \tilde{u}_j^t)|_{\partial\Omega_0}, f_{2,j} := \partial(u'_j - \tilde{u}_j^t)/\partial\nu|_{\partial\Omega_0}, k^2 = k_1^2, nk^2 = k_2^2$  and  $\text{Im } n \geq 0$ . It is clear that  $f_{1,j} = f_{2,j} = 0$  on  $\Gamma^* = \Gamma \cap \partial\Omega_0$ . Since  $z^*$  has a positive distance from  $\tilde{\Gamma}$ , it follows from 4.2 (ii) that

$$\|\tilde{u}'_j\|_{L^2(\Omega_0)} + \|\tilde{u}'_j\|_{H^{\frac{1}{2}}(\partial\Omega_0 \setminus \Gamma^*)} + \left\| \frac{\partial \tilde{u}'_j}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega_0 \setminus \Gamma^*)} \leq C$$

uniformly with respect to  $j \in \mathbb{N}$ . Let  $K = \Omega_0 \setminus B_{2\delta}(z^*)$ . Then  $\text{dist}(K, B_\delta(z^*)) = \delta$ . Theorem 4.1 implies that  $\|u'_j\|_{H^1(K)} \leq C$  uniformly with respect to  $j \in \mathbb{N}$ . This, together with the trace theorem, implies that

$$\|u'_j\|_{H^{\frac{1}{2}}(\partial\Omega_0 \setminus \Gamma^*)} + \left\| \frac{\partial u'_j}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega_0 \setminus \Gamma^*)} \leq C$$

uniformly with respect to  $j \in \mathbb{N}$ . Now we can invoke Theorem 5.1 to conclude that

$$\|\tilde{u}_j^t\|_{L^2(\Omega_0)} \leq C \quad (5.7)$$

uniformly with respect to  $j \in \mathbb{N}$ . In fact if  $\text{Im}(n) > 0$ , then according to Theorem 5.1 (i) the constructed interior transmission problem on  $\Omega_0$  is well-posed. On the other hand, if  $\text{Im}(n) =$

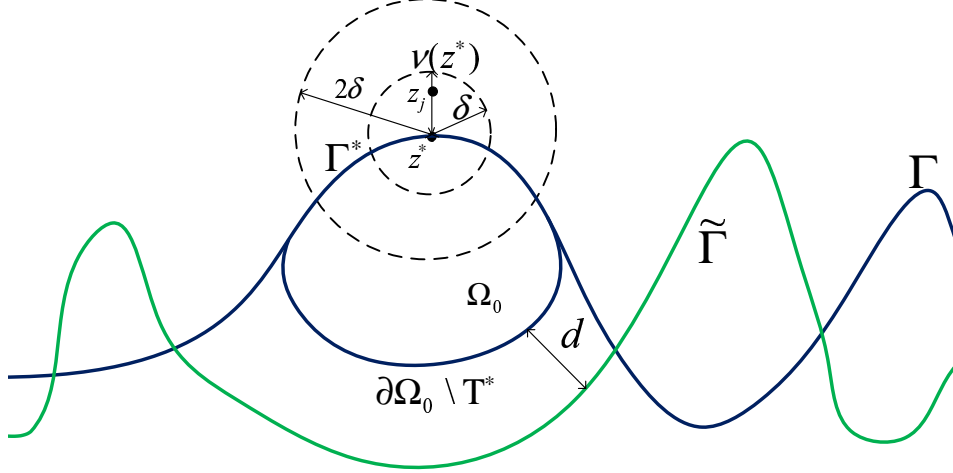


Figure 2: Geometry in Step 1.

0, then by Theorem 5.1 (ii) we can choose  $\Omega_0$  sufficiently small so that  $k^2$  is not an interior transmission eigenvalue on  $\Omega_0$ . In either case of above, (5.7) follows from the estimate (5.1).

Notice that the scattered field  $\tilde{u}'_j$  is bounded uniformly with respect to  $j \in \mathbb{N}$ , so we get the estimate  $\|G'_{k_1}(\cdot, z_j)\|_{L^2(\Omega_0)} \leq C$  uniformly with respect to  $j \in \mathbb{N}$ . This is a contradiction since  $G'_{k_1}(\cdot, z^*)$  is not locally integrable in  $\Omega_0$ . Thus we have  $\Gamma = \tilde{\Gamma}$ .

**Step 2.** We show that  $D = \tilde{D}$ .

By **Step 1**, we already have  $\Omega_1 = \tilde{\Omega}_1$ . Let  $S$  be the unbounded connected component of  $\mathbb{R}^2 \setminus \{\overline{\Omega_1} \cup D \cup \tilde{D}\}$ . For  $z \in S$ , we claim that

$$u(x; z) = \tilde{u}(x; z) \quad \text{for all } x \in \overline{S} \quad (5.8)$$

In fact, for  $z \in \Omega_1$ , by Step 1, we have  $u^t(x; z) = \tilde{u}^t(x; z)$  for all  $x \in \overline{\Omega_1}$  and  $x \neq z$  and by the transmission conditions we obtain that

$$u(x; z) = \tilde{u}(x; z), \quad \frac{\partial u}{\partial \nu}(x; z) = \frac{\partial \tilde{u}}{\partial \nu}(x; z) \quad \text{for all } x \in \Gamma, z \in \Omega_1$$

Since  $u$  and  $\tilde{u}$  satisfy the Helmholtz equation in  $S$ , Holmgren's uniqueness theorem implies that

$$u(x; z) = \tilde{u}(x; z) \quad \text{for all } x \in \overline{S}, z \in \Omega_1$$

This together with the reciprocity relation of the total field, implies that  $u(z; x) = \tilde{u}(z; x)$  for all  $x \in S, z \in \Omega_1$ . Regarding  $u$  and  $\tilde{u}$  as functions of  $z$  and repeating the same argument as above yield that  $u(z; x) = \tilde{u}(z; x)$  for all  $x, z \in S$ . Since the scattered fields are continuous up to the boundary, by exchanging  $z$  and  $x$ , (5.8) holds.

Assume that  $D \neq \tilde{D}$ . Without loss of generality, we may assume that there exists  $z^* \in \partial D \setminus \partial \tilde{D}$ . Define  $z_j := z^* + \delta/j\nu(z^*)$ ,  $j \in \mathbb{N}^+$ , with  $\delta > 0$  such that  $z_j \in B_\delta(z^*)$  and  $\overline{B_\delta(z^*)} \cap \tilde{D} = \emptyset$ . Since there is a positive distance between  $B_\delta(z^*)$  and  $\tilde{D}$ , by (5.8) and Remark 4.2 (ii), it follows that

$$\left\| \frac{\partial u(\cdot; z_j)}{\partial \nu} + i\beta u(\cdot; z_j) \right\|_{H^{-\frac{1}{2}}(\partial \Gamma_2 \cup B_\delta(z^*))} + \|u(\cdot; z_j)\|_{H^{\frac{1}{2}}(\partial \Gamma_1 \cup B_\delta(z^*))} \leq C,$$

where the constant  $C > 0$  is independent of  $j$ . But  $u(\cdot; z_j)$  satisfies boundary conditions, so

$$\begin{aligned} & \left\| \frac{\partial u(\cdot; z_j)}{\partial \nu} + i\beta u(\cdot; z_j) \right\|_{H^{-\frac{1}{2}}(\partial\Gamma_2 \cup B_\delta(z^*))} + \|u(\cdot; z_j)\|_{H^{\frac{1}{2}}(\partial\Gamma_1 \cup B_\delta(z^*))} = \\ & \left\| \frac{\partial G_{k_2}(\cdot; z_j)}{\partial \nu} + i\beta G_{k_2}(\cdot; z_j) \right\|_{H^{-\frac{1}{2}}(\partial\Gamma_2 \cup B_\delta(z^*))} + \|G_{k_2}(\cdot; z_j)\|_{H^{\frac{1}{2}}(\partial\Gamma_1 \cup B_\delta(z^*))} \\ & \rightarrow \infty \quad \text{as } j \rightarrow \infty \end{aligned}$$

This is a contradiction, which means that  $D = \tilde{D}$ .

**Step 3.** We show that the physical property is uniquely determined, that is,  $\mathcal{B} = \tilde{\mathcal{B}}$ . First, as a result of **Step 2**, we claim that

$$\Gamma_i = \tilde{\Gamma}_i, \quad i = 1, 2 \quad (5.9)$$

In fact, suppose (5.9) is not true. Then  $\Gamma_1 \cap \tilde{\Gamma}_2 \neq \emptyset$ . For  $z \in \Sigma_s$  we have  $u(\cdot; z) = \partial u(\cdot; z)/\partial \nu = 0$  on  $\Gamma_1 \cap \tilde{\Gamma}_2$ , and by Holmgren's uniqueness theorem,  $u(\cdot; z) = 0$  in  $\Omega_2$ . Thus,  $u^t(\cdot; z) = \partial u^t(\cdot; z)/\partial \nu = 0$  on  $\Gamma$ . Applying Holmgren's uniqueness theorem again, we have  $u(\cdot; z) = G_{k_1}(\cdot; z)$  in  $\Omega_1 \setminus B_\delta(z)$  for any  $\delta > 0$  such that  $\overline{B_\delta(z)} \cap \Gamma = \emptyset$ . Let  $\delta \rightarrow 0$  to get that  $\|u(x; z)\|_{H^1(B_\delta(z))} \rightarrow \infty$ , which contradicts to Remark 4.2 (ii). Thus, (5.9) holds.

Next, we may assume that  $\Gamma_1$  and  $\tilde{\Gamma}_1$  are both nonempty. If the impedance function  $\beta \neq \tilde{\beta}$ , then from the boundary condition on  $\Gamma_1$

$$\frac{\partial u(\cdot; z)}{\partial \nu} + i\beta u(\cdot; z) = 0, \quad \frac{\partial u(\cdot; z)}{\partial \nu} + i\tilde{\beta} u(\cdot; z) = 0 \quad \text{on } \Gamma_1, \text{ for } z \in \Sigma_s,$$

which gives

$$(\beta - \tilde{\beta})u(\cdot; z) = 0 \quad \text{on } \Gamma_1 \text{ for } z \in \Sigma_s$$

Consequently,  $\partial u(\cdot; z)/\partial \nu = u(\cdot; z) = 0$  on the open set  $\{x \in \partial D : \beta(x) \neq \tilde{\beta}(x)\}$ . Then, we get the same contradiction as that in proving (5.9). Hence,  $\mathcal{B} = \tilde{\mathcal{B}}$ . The proof is thus finished.  $\square$

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